ELSEVIER Journal of Pure and Applied Algebra 119 (1997) 171–183

JOURNAL OF PURE AND APPLIED ALGEBRA

Second analytic deviation one ideals and their Rees algebras

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Communicated by C.A. Weibel; received 6 December 1994; revised 25 October 1995

Abstract

We study the Cohen–Macaulayness and the defining equations of Rees algebras of ideals with good residual intersection properties, especially for ideals having second analytic deviation one. In particular we show that, under certain conditions, the Cohen–Macaulayness of the Rees algebra forces the ideal to have the expected reduction number. © 1997 Elsevier Science B.V.

1. Introduction

Let R be a Noetherian ring and let I be an R-ideal. There is a canonical morphism

$$\alpha : S(I) \to R[It]$$

from the symmetric algebra of I onto the Rees algebra $R[It] = \bigoplus_{j \geq 0} I^j$. The symmetric algebra of I is defined by the linear relations on I, which are determined by giving a presentation matrix of I. The ideal I is said to be of *linear type* when α is an isomorphism. Otherwise we define the *relation type* of I, $rt(I)$, to be the maximal degree of a minimal generator of $\mathcal{A} = \ker \alpha$.

One may study the Rees algebra more intrinsically by considering reductions. Suppose that R is local with infinite residue field k. An ideal $J \subset I$ is a *reduction* of I if the extension of Rees algebras $R[Jt] \subset R[It]$ is finite, or equivalently if $I^{r+1} = JI^r$ for some integer $r \geq 0$. Denote the least such r by $r_J(I)$. A reduction J is a *minimal reduction* of I if it is minimal with respect to inclusion among all reductions of I. The

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0022-4049/97/$17.00
PII S0022-4049(96)00021-7

reduction number of I is then defined by $r(I) = \min\{r_J(I) \mid J$ is a minimal reduction of $I\}$. An important invariant is the *analytic spread* $\ell(I)$, which is the minimal number of generators $v(J)$ of any minimal reduction J of I, or equivalently the dimension of the fiber ring $R[It] \otimes_R k$ [15].

If I is of linear type, then in particular $v(I) = \ell(I)$, or in other words I is generated by analytically independent elements. In general, it is useful to define the *second analytic deviation* to be the nonnegative integer $v(I) - \ell(I)$. (If I has grade g, the *analytic deviation* [10] is $\ell(I) - g$, while the *deviation* is $v(I) - g$.) In Theorem 4.3, we show that there exist perfect prime ideals which are not of linear type, but are locally generated by analytically independent elements (answering a question of [19]). However, the second analytic deviation zero case is mostly well-understood for various classes of ideals, such as the class of strongly Cohen–Macaulay ideals (cf. [8]). Recall that an R-ideal is *strongly Cohen–Macaulay* if the Koszul homology modules H_i built on a generating set of I are Cohen–Macaulay. The first nontrivial case to study next is when I has second analytic deviation one.

We will need to use some termininology of [20]. First, recall that if s is an integer, we say that I satisfies G_s if $v(I_p) \leq \dim R_p$ for all $p \in V(I)$ with $\dim R_p \leq s-1$. We say that I satisfies G_∞ if I satisfies G_s for every s.

Definition 1.1. Let R be a local Cohen–Macaulay ring, let I be an R-ideal and let s be an integer.

(a) A proper R-ideal K is an s-residual intersection of I if $s \geq ht\ I$ and there is an R-ideal $\mathfrak{a} \subset I$, such that $K = \mathfrak{a} : I$ and $ht\ K \geq s \geq v(\mathfrak{a})$. The s-residual intersection K is a geometric s-residual intersection if in addition $ht\ I + K > s$.

(b) We say I satisfies AN_s^- if for every $g \leq i \leq s$ and every geometric i-residual intersection K of I, R/K is Cohen–Macaulay.

The properties AN_s^-, called Artin-Nagata properties, are closely related to the Cohen–Macaulayness of Rees algebras [9, 12, 13, 20]. In the sequel, we will denote by $\mathscr{R}$ the Rees algebra of I, and by G the associated graded ring $gr_I R = \bigoplus_{j \geq 0} I^j/I^{j+1} \cong \mathscr{R} \otimes_R R/I$. We recall the following result on the Cohen–Macaulayness of the Rees algebra, which was one of the main results of [13]:

Theorem 1.2 [13, 3.1]. *Let R be a local Cohen–Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g, analytic spread ℓ and reduction number r, let $k \geq 1$ be an integer with $r \leq k$ and assume that I satisfies G_ℓ and $AN_{\ell-3}^-$ locally in codimension $\ell - 1$, that I satisfies $AN_{\ell-\max\{2,k\}}^-$, and that depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq k$. Then G is Cohen–Macaulay, and $\mathscr{R}$ is Cohen–Macaulay if $g \geq 2$.*

Knowing that these algebras are Cohen–Macaulay, it is natural to ask about the nature of their defining equations. Generalizing the result [13, 4.10], we are able to compute the number and degrees of the defining equations of the Rees algebra:

Theorem 2.6. *In addition to the assumptions of Theorem* 1.2, *assume that* I *satisfies* $AN^-_{\ell-2}$ *and that* $S_j(I) \cong I^j$ *for* $1 \le j \le r$. *Then* $\mathscr{A}$ *is minimally generated by* $\binom{n-\ell+r}{n-\ell-1}$ *forms of degree* $r+1$.

For a perfect Gorenstein ideal of grade 3, having second analytic deviation one and reduction number $r \le \ell - g + 1$, we are able to explicitly describe the equations of its Rees algebra (Theorem 2.10). They are obtained in a straightforward way from a Jacobian dual of the presentation matrix, which is analagous, but not identical, to the grade 2 case [23]. After the work of this paper was completed we learned that a similar result was proved by Morey [14] when the presentation matrix has linear entries.

We are also able to show a partial converse of Theorem 1.2. Similar results have been obtained recently in [1–3] and by different methods in [22] and [17].

Theorem 3.8. *Let* R *be a local Gorenstein ring with infinite residue field, let* I *be a strongly Cohen–Macaulay* R*-ideal of grade* $g \ge 2$, *analytic spread* ℓ, *and minimal number of generators* $n = \ell + 1$, *assume that* I *satisfies* G_ℓ, *and that* $I \subset I_1(\phi)^2$, *where* ϕ *is a matrix with* n *rows presenting* I. *Then the following are equivalent*:

(a) *After elementary row operations, the entries of one row generate* $I_1(\phi)$.

(b) $r(I) \le \ell - g + 1$.

(c) $rt(I) \le \ell - g + 2$.

(d) $\mathscr{A}$ *is cyclic.*

(e) $\mathscr{R}$ *is Cohen–Macaulay.*

(f) G *is Cohen–Macaulay.*

The equivalence of (e) and (f) with (a)–(d) is the new information in this result. The equivalence of (a)–(d), and that these imply (e) was proved by Simis et al. in [17] (without the condition that $I \subset I_1(\phi)^2$). They also proved that (e) is equivalent to (a)–(d) when $g = 2$. The first instance of these results, especially the *row condition* (a), was studied by Aberbach and Huneke in [3]. In particular, by the structure theorem of [6] we obtain the equivalences in Theorem 3.8 for any perfect Gorenstein ideal of grade 3 satisfying G_ℓ and having second analytic deviation one.

2. Number of defining equations

We will make use of the following remark.

Remark 2.1 ([20, 1.6], [10, 3.2], [17, proof of 3.4]). Let R be a local Cohen–Macaulay ring with infinite residue field, and let be I an R-ideal of grade g, analytic spread ℓ and assume that I satisfies G_ℓ. Then there exists a minimal reduction J of I and a generating set $a_1, \ldots, a_\ell$ of J with the following property: J is an ℓ-residual intersection of I (or $J = I$), and for $\mathfrak{a}_i = (a_1, \ldots, a_i)$ with $g \le i \le \ell - 1$, $\mathfrak{a}_i : I$ is a geometric

i-residual intersection of I. Moreover, every permutation of $a_1,\dots,a_\ell$ enjoys the same property.

Once and for all we will fix the choice of a minimal reduction J, and generating sets $J=(a_1,\dots,a_\ell)$, $I=(a_1,\dots,a_\ell,\dots,a_n)$ and the ideals $\mathfrak{a}_i=(a_1,\dots,a_i)$ as in the above remark.

We will need several lemmas, all of which are essentially known in somewhat less generality.

Lemma 2.2. *Let R be a local Cohen–Macaulay ring with infinite residue field, let I be an R-ideal of grade g, analytic spread ℓ, assume that I satisfies G_ℓ, let $\mathfrak{a}_i$ be the ideals defined in Remark* 2.1, *and assume that G is Cohen–Macaulay. Then $(\mathfrak{a}_i:(a_{i+1}))\cap I^j=\mathfrak{a}_iI^{j-1}$ for $0\le i\le \ell-1$ and $j\ge i-g+1$.*

Proof. By [3, 5.9 and 5.10] and Remark 2.1 one has the equation

$$((\mathfrak{a}_i+I^{j+2}):(a_{i+1}))\cap I^j=\mathfrak{a}_iI^{j-1}+I^{j+1}$$

whenever $0\le i\le \ell-1$ and $j\ge i-g+1$. But then $(\mathfrak{a}_i:(a_{i+1}))\cap I^j\subset \mathfrak{a}_iI^{j-1}+I^{j+1}$. Since $\bigcap_{k\ge 1}I^{j+k}=0$, the result follows. □

We will need the following result which complements Lemma 2.2 in low degrees.

Lemma 2.3 [13, 2.5]. *Let R be a local Cohen–Macaulay ring of dimension d with infinite residue field, let k and s be integers, let I be an R-ideal of analytic spread ℓ and assume that I satisfies G_ℓ and $AN_{\ell-3}$ locally in codimension $\ell-1$, that I satisfies AN_s^-, that depth $R/I^j\ge d-\ell+k-j$ for $1\le j\le k$, and let $\mathfrak{a}_i$ be the ideals as in Remark* 2.1. *Then $(\mathfrak{a}_i:(a_{i+1}))\cap I^j=\mathfrak{a}_iI^{j-1}$ for $0\le i\le \ell-1$ and $\max\{1,i-s\}\le j\le k$.*

Let $S=R[T_1,\dots,T_n]$ be a polynomial ring over R, and present the Rees algebra as $\mathscr{R}\cong S/Q$, by mapping T_i to $a_it\in R[It]$.

Lemma 2.4. *Let R be a local Cohen–Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with grade g and analytic spread ℓ, assume that I satisfies G_ℓ, let $k\ge 1$ be an integer and assume one of the following conditions holds:*

(a) $k\ge \ell-g$ and G is Cohen–Macaulay.

(b) I satisfies $AN_{\ell-3}^-$ locally in codimension $\ell-1$, I satisfies $AN_{\ell-k-1}^-$, and depth $R/I^j\ge d-\ell+k-j$ for $1\le j\le k$.

Then

$$[(T_1,\dots,T_\ell)\cap Q]_{k+1}\subset[Q]_kS.$$

Proof [2, 2.1]. It will be enough to prove $[(T_1,\dots,T_i)\cap Q]_{k+1}\subset[Q]_kS$ for $0\le i\le \ell$, which we do by induction on i. Since $i=0$ is trivial, we may assume $i\ge 1$ and that the result holds for smaller i. Let $F \in (T_1,\dots,T_i) \cap Q$ be a form of degree $k+1$ and write $F = \sum_{j=1}^{i} G_j T_j$ where $G_j \in [S]_k$. Evaluation gives $0 = \sum_{j=1}^{i} G_j(a_1,\dots,a_n)a_j$, hence $G_i(\underline{a}) \in (\mathfrak{a}_{i-1} : (a_i)) \cap I^k = \mathfrak{a}_{i-1}I^{k-1}$ by Lemma 2.2 in case (a) or by Lemma 2.3 in case (b). Hence there are forms $H_1,\dots,H_{i-1} \in [S]_{k-1}$ for which $P = G_i - \sum_{j=1}^{i-1} H_j T_j \in [Q]_k S$. But then $F - T_i P = \sum_{j=1}^{i-1}(G_j + T_i H_j)T_j \in [(T_1,\dots,T_{i-1}) \cap Q]_{k+1} \subset [Q]_k S$ by induction, hence $F \in [Q]_k S$. □

Proposition 2.5. *Let R be a local Cohen–Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ and reduction number r, assume that I satisfies G_ℓ and $AN^-_{\ell-3}$ locally in codimenson $\ell - 1$, that $S_j(I) \cong I^j$ and depth $R/I^j \geq d - \ell + k - j$ for $1 \leq j \leq r$, and that I satisfies $AN^-_{\ell-r-1}$. Then $\nu([\mathscr{A}]_{r+1}) = \binom{n-\ell+r}{n-\ell-1}$.*

Proof. Consider the exact sequence

$$0 \to \mathscr{A} \to S(I) \to \mathscr{R}(I) \to 0$$

which in degree $r+1$ is

$$0 \to [\mathscr{A}]_{r+1} \to S_{r+1}(I) \to I^{r+1} \to 0.$$

Now if $r = 0$ then $n = \ell$, and $\mathscr{A} = 0$ by [20, 1.11]. Hence we may assume that $r > 0$. Since we may present $\mathscr{R} \cong S/Q$ and $S(I) \cong S/L$, where L is the ideal generated by the linear forms in Q, we have $\mathscr{A} \cong Q/L$. The latter exact sequence will induce an exact sequence

$$0 \to [\mathscr{A}]_{r+1} \to S_{r+1}(I/J) \to I^{r+1}/JI^r \to 0$$

once we have have shown that $[(L,T_1,\dots,T_\ell) \cap Q]_{k+1} \subset L$, or equivalently that $[(T_1,\dots,T_\ell) \cap Q]_{k+1} \subset L$. But this is clear from Lemma 2.4(b) since by the assumption on the symmetric powers we have $[Q]_k S \subset [Q]_1 S = L$. As $I^{r+1}/JI^r = 0$, we conclude that $[\mathscr{A}]_{r+1} \cong S_{r+1}(I/J)$ from which the result is immediate. □

This proposition allows us to compute the number of defining equations for the Rees algebras of ideals having the *minimal reduction number*. We take this to mean that $S_j(I) \cong I^j$ for $1 \leq j \leq r$ (and $r > 0$), or equivalently that $\mathscr{A}_{r+1}$ is the first nonvanishing component of $\mathscr{A}$, where $r = r(I)$ is the reduction number.

Theorem 2.6. *Let R be a local Cohen–Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ, minimal number of generators n, and reduction number r and assume that I satisfies G_ℓ and $AN^-_{\ell-2}$, and that $S_j(I) \cong I^j$ and depth $R/I^j \geq d - \ell + r - j$ for $1 \leq j \leq r$. Then $\mathscr{A}$ is minimally generated by $\binom{n-\ell+r}{n-\ell-1}$ forms of degree $r+1$.*

Proof. By [13, 4.6] we have $rt(I) \leq r+1$. The result now follows from Proposition 2.5. □

Corollary 2.7 [13, 4.10]. *Let R be a local Cohen–Macaulay ring with infinite residue field, let I be a strongly Cohen–Macaulay R-ideal of grade g, with analytic spread ℓ and minimal number of generators n, and assume that I satisfies G_ℓ and that $r(I) \leq \ell - g + 1$. Then $\mathscr{A}$ is minimally generated by $\binom{n-g+1}{\ell-g+2}$ forms of degree $\ell - g + 2$.*

Proof. Since I is strongly Cohen–Macaulay, it satisfies AN_ℓ^- by e.g. [9, 3.3], and by [8, proof of 4.6] $S_j(I) \cong I^j$ whenever $1 \leq j \leq \ell - g + 2$. The result now follows from Theorem 2.6. □

Corollary 2.8. *Let R be a local Cohen–Macaulay ring with infinite residue field, and let I be an R-ideal with analytic spread ℓ, minimal number of generators n, and assume that I satisfies $G_{\ell+1}$ and $AN_{\ell-2}^-$, and that depth $R/I \geq d - \ell$. Then $\mathscr{A}$ is minimally generated by $\binom{n-\ell+1}{2}$ quadrics.*

Proof. Since I satisfies $AN_{\ell-2}^-$, it follows by [20, 1.8] that $\nu(I_p) = \ell(I_p)$ whenever $\dim R_p = \ell$. Hence by [20, 4.1] or [13, 4.7] it follows that $r(I) \leq 1$, and the result thus follows from Theorem 2.6. □

The condition $AN_{\ell-2}^-$ is somewhat harmless in small analytic deviation. In analytic deviation one it is vacuous, while in analytic deviation two, it is equivalent to Cohen–Macaulayness (at least if I is unmixed and R is Gorenstein (e.g. [20])). Thus one obtains:

Corollary 2.9. *Let R be a local Gorenstein ring with infinite residue field and let I be an R-ideal with grade g, analytic spread ℓ and assume that I satisfies $G_{\ell+1}$.*

(a) *If $\ell = g + 1$ and depth $R/I \geq \dim R/I - 1$ then $\mathscr{A}$ is minimally generated by $\binom{n-g}{2}$ quadrics.*

(b) *If $\ell = g + 2$ and R/I is Cohen–Macaulay, then $\mathscr{A}$ is minimally generated by $\binom{n-g-1}{2}$ quadrics.*

Now let R be a local Gorenstein ring with infinite residue field, and let I be a strongly Cohen–Macaulay ideal satisfying G_ℓ, having second analytic deviation one and the expected reduction number $\ell - g + 1$. By [17] (or Theorem 1.2 and Corollary 2.7) the Rees algebra $\mathscr{R}$ is Cohen–Macaulay and $\mathscr{A}$ is generated by a single form in degree $\ell - g + 2$. We now consider the problem of computing the single generator of $\mathscr{A}$ *explicitly*. This has been studied when I is a perfect ideal of grade 2 by Vasconcelos [23].

Let

$$R^m \xrightarrow{\phi} R^n \to I \to 0$$

be a minimal presentation of I, where ϕ is a matrix with n rows. Recall that the symmetric algebra has a presentation $S(I) \cong R[T_1, \ldots, T_n]/(\ell_1, \ldots, \ell_m)$ defined by the equations

$$(\ell_1, \ldots, \ell_m) = (T_1, \ldots, T_n)\phi.$$

It also follows from [17] (or Proposition 3.2) that ϕ satisfies the row condition: after elementary row operations, the entries of $I_1(\phi)$ can be generated by the last row of ϕ. Moreover, it is known that $I_1(\phi)$ is Gorenstein [17, 4.13] and has height ℓ [8, 9.1], but let us assume further that it is a complete intersection and choose a generating set $I_1(\phi) = (x_1,\dots,x_\ell)$. Now we may assume after row and column operations that $I_1(\phi)$ is generated by the entries of the last row from among the first ℓ columns. Let ψ be the ℓ by ℓ submatrix of ϕ obtained by deleting the last row and the last $m-\ell$ columns, and over the polynomial ring $R[T_1,\dots,T_\ell]$ consider a Jacobian dual [16] of ψ:

$$(T_1,\dots,T_\ell)\psi = (x_1,\dots,x_\ell)B(\psi).$$

Here $B = B(\psi)$ is an ℓ by ℓ matrix whose entries are linear forms in the variables $T_1,\dots,T_\ell$. The characteristic polynomial of B determines a relation on the Rees algebra.

Now in particular, let I be a perfect Gorenstein ideal of grade 3 satisfying G_ℓ and the row condition. By the structure theorem of [6], ϕ may be taken as an alternating n by n matrix, and that $I_1(\phi)$ is generated by the last row of the ϕ. Since ϕ is alternating, it follows that $I_1(\phi)$ is automatically a complete intersection.

Theorem 2.10. *Let R be a local Gorenstein ring with infinite residue field, let I be a perfect Gorenstein ideal of grade 3, with analytic spread ℓ, minimal number of generators $n = \ell + 1$, and assume that I satisfies G_ℓ, let ϕ be an n by n alternating matrix presenting I with last row $(-x_1,\dots,-x_\ell,0)$ which generates the ideal of entries of ϕ and let ψ be the ℓ by ℓ alternating submatrix of ϕ obtained by deleting the last row and column. Then there exists a Jacobian dual $B = B(\psi)$:*

$$(T_1,\dots,T_\ell)\psi = (x_1,\dots,x_\ell)B(\psi)$$

such that $\mathscr{A}$ is generated by $F = T_n^{-1}\chi_B(T_n)$, where $\chi_B(T_n)$ denotes the characteristic polynomial of B in the variable T_n. Moreover, if we let ψ_j denote the jth column of ψ, for $1 \le j \le \ell$, and write $\psi_j = A_j(\underline{x})^t$, where A_j is an ℓ by ℓ matrix whose jth row consists of zeros and whose ith row, for any $1 \le i \le \ell$, is the negative of the jth row of A_i, then B may be taken to be the matrix whose jth column is $A_j^t(\underline{T})^t$.

Proof. It is enough to prove the second statement. Let a_{ij} and b_{ij} be the ijth entry of the matrices ψ and B respectively, and let a_{ijk} be the ikth entry of A_j. Then write $a_{ij} = \sum_{k=1}^{\ell} a_{ijk}x_k$ and $b_{ij} = \sum_{k=1}^{\ell} a_{kji}T_k$, where $a_{ijk} = -a_{jik}$ for $i \neq j$ and $a_{ijk} = 0$ for $i = j$. Let $\{z_{ijk}\}$ be a new set of variables with $z_{ijk} = -z_{jik}$ for $i \neq j$ and $z_{ijk} = 0$ for $i = j$, and define $\tilde{a}_{ij} = \sum_{k=1}^{\ell} z_{ijk}x_k$ and $\tilde{b}_{ij} = \sum_{k=1}^{\ell} z_{kji}T_k$. Denote by $\tilde{\psi}$ and $\tilde{B}$ the matrices whose ijth entry is $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$, respectively. It follows that $\tilde{B}$ is a Jacobian dual of $\tilde{\psi}$:

$$(T_1,\dots,T_\ell)\tilde{\psi} = (x_1,\dots,x_\ell)\tilde{B}. \tag{2.11}$$

Now it will be enough to show that $\det(\tilde{B}) = 0$. For then by specializing, it holds that $\det(B) = 0$, and hence that T_n divides the characteristic polynomial $\chi_B(T_n)$ of B.

Since the characteristic polynomial may be obtained as a minor of a Jacobian dual of ϕ, it is clear that $\chi_B(T_n)$ is a relation on the Rees algebra. But by Corollary 2.7, $\mathscr{A}$ is cyclic, hence since the quotient $F = T_n^{-1}\chi_B(T_n)$ is monic, it is the required form of degree $\ell - 1 = \ell - g + 2$.

Now to show the claim, multiply Eq. (2.11) on the right by the column $(T_1, \dots, T_\ell)^t$. Since $\tilde{\psi}$ is an alternating matrix,

$$(\underline{x})\tilde{B}(T_1, \dots, T_\ell)^t = 0.$$

Since the x's form a regular sequence, the entries of the matrix $\tilde{B}(T_1, \dots, T_\ell)^t$ belong to the ideal generated by $(x_1, \dots, x_\ell)$ and the subring over k generated by the T's and the z's. It follows that $\tilde{B}(T_1, \dots, T_\ell)^t = 0$ and hence that $\det(\tilde{B}) = 0$. □

Example 2.12. Let $I \subset k[[x, y, z, w]]$ be the defining ideal of the Gorenstein monomial curve $k[[t^5, t^6, t^7, t^8]]$. Then I has a presentation matrix ϕ:

$$\begin{pmatrix} 0 & z & w & y & y \\ -z & 0 & x^2 - w & w - y & w \\ -w & w - x^2 & 0 & 0 & z \\ -y & y - w & 0 & 0 & x \\ -y & -w & -z & -x & 0 \end{pmatrix}.$$

Since $I_1(\phi) = (x, y, z, w)$, ϕ satisfies the row condition. Deleting the last row and column, we obtain the Jacobian dual B :

$$\begin{pmatrix} -T_4 & T_4 & 0 & T_1 - T_2 \\ -T_3 & T_3 - T_4 & T_1 - T_2 & T_2 \\ -T_2 & T_1 & 0 & 0 \\ 0 & -xT_3 & xT_2 & 0 \end{pmatrix}.$$

Dividing the characteristic polynomial $\chi_B(T_5)$ by T_5 gives us the nontrivial cubic relation on the Rees algebra:

$$T_5^3 + 2T_4T_5^2 + T_4^2T_5 + T_1T_2T_5 - T_1^2T_5 - T_2T_4^2 - T_2^2T_4 + 2T_1T_2T_4$$
$$-T_1^2T_4 + xT_2T_3T_5 + xT_2T_3T_4 + xT_2T_3^2 - xT_1T_3^2 - xT_2^3.$$

3. Expected reduction number

In this section we now restrict to the case when I has second analytic deviation one, and attempt to give converses to Theorem 1.2 asserting that I has a "small" reduction number.

The next result follows exactly as in [2], but we repeat the proof for convenience.

Lemma 3.1. *With the assumptions of Lemma 2.4, assume in addition $n = \ell + 1$, and let ϕ be a minimal presentation matrix of I. Then $I_1(\phi)[Q]_{k+1} \subset [Q]_kS$.*

Proof. Let $F \in [Q]_{k+1}$ and write $F = \alpha T_n^{k+1} + G$ where $G \in [(T_1,\ldots,T_\ell)]_j$ and $\alpha \in R$. Since Lemma 2.4 clearly holds for any permutation of $T_1,\ldots,T_n$ (by Remark 2.1) and $x \in I_1(\phi)$, we may assume $x \in (a_1,\ldots,a_{n-1}):(a_n)$. Hence there is a linear form $H = xT_n + \sum_{i=1}^{\ell} r_iT_i \in [Q]_1, r_i \in R$. But then $xF = \alpha xT_n^{k+1} + xG = \alpha T_n^k H - \alpha T_n^k(H - xT_n) + xG \in Q_1 + [(T_1,\ldots,T_\ell) \cap Q]_{k+1} \subset [Q]_k S$ by Lemma 2.4. □

The following shows that the row condition is satisfied, under mild conditions, when I has second analytic deviation one with the minimal reduction number.

Proposition 3.2. *Let R be a local Cohen–Macaulay ring of dimension d with infinite residue field, let I be an R-ideal with analytic spread ℓ, minimal number of generators $n = \ell + 1$ and reduction number r, assume that I satisfies G_ℓ and $AN_{\ell-3}^-$ locally in codimension $\ell - 1$, that I satisfies $AN_{\ell-r-1}^-$, that $S_j(I) \cong I^j$ and depth $R/I^j \geq d - \ell + r - j$ for $1 \leq j \leq r$, and let ϕ be a matrix presenting I with n rows. Then, after elementary row operations, the entries of one row generate $I_1(\phi)$.*

Proof. Let J be a minimal reduction as in Remark 2.1 and put $K = J : I = J : (a_n)$, which is, after elementary row operations, the ideal generated by the last row of ϕ. Hence it is enough to show that $K = I_1(\phi)$.

By Lemma 3.1, $I_1(\phi)[Q]_{r+1} \subset [Q]_r S = [Q]_1 S$ by assumption on the symmetric powers (note that $r \geq 1$). Let $F \in [Q]_{r+1}$ with $F = T_n^{r+1} + G$ with $G \in [(T_1,\ldots,T_\ell)]_{r+1}$. Let $x \in I_1(\phi)$. Then $xF \in [Q]_1 S$ and hence there is an equation $xF = xT_n^{r+1} + xG = \sum_{i=1}^m L_iH_i$ where $L_i \in [Q]_1$ and $H_i \in [S]_r$. But by comparing the coefficients of the term T_n^{r+1} it is clear that $x \in K$. □

We are now ready to prove one of our main technical results, which has been shown in [2] and [1] for ideals having small analytic deviation.

Theorem 3.3. *Let R be a local Cohen–Macaulay ring of dimension d with infinite residue field, let I be an R-ideal of grade g, analytic spread ℓ, and minimal number of generators $n = \ell + 1$, assume that I satisfies G_ℓ, that G is Cohen–Macaulay, that $I \subset I_1(\phi)^e$ for some $e \geq 2$ and minimal presentation matrix ϕ, let $k \geq 1$ be an integer such that $S_j(I) \cong I^j$ for $1 \leq j \leq k$, and further assume that one of the following conditions holds:*

(a) *$k \geq \ell - g$.*

(b) *depth $R/I^j \geq d - g - j - 1$ for $1 \leq j \leq \ell - g - 2$, I satisfies $AN_{\ell-k-1}^-$, and I satisfies $AN_{\ell-3}^-$ locally in codimension $\ell - 1$.*

Then $r(I) \leq \ell - g + 1 + (\ell - g + 1 - k)/(e - 1)$.

Proof. Let $r = r(I) \geq k$ and choose a form $F \in [Q]_{r+1}$ with $F = T_n^{r+1} + G$ where $G \in [(T_1,\ldots,T_\ell)]_{r+1}$. Since $F \in [Q]_{r+1}$, repeated application of Lemma 3.1 gives $I_1(\phi)^{r-k+1}F \in [Q]_k S$. (Note that in case (b) the condition on the depth of $R/I^{\ell-g-1}$ is satisfied by [7, 3.3] as G is Cohen–Macaulay.) By the assumption on the symmetric powers we have $[Q]_k S \subset [Q]_1 S$, hence $I_1(\phi)^{r-k+1}F \in [Q]_1 S$. Now as in the proof of Proposition 3.2, it follows that $I_1(\phi)^{r-k+1} \subset K$. Let $\xi = \lceil (r-k+1)/(e) \rceil$. Then by the assumption $I \subset I_1(\phi)^e$, we have $I^{\xi+1} = I^{\xi} I \subset (I_1(\phi)^e)^{\xi} I \subset I_1(\phi)^{r-k+1} I \subset KI = (J : I)I \subset J$. In particular, $I^{\xi+\ell-g+1} \subset JI^{\ell-g}$. The result now follows from the next lemma. □

Lemma 3.4. *Let R be a local Cohen–Macaulay ring with infinite residue field, and let I be an R-ideal with analytic spread ℓ and assume that I satisfies G_ℓ, that G is Cohen–Macaulay, that J is a minimal reduction of I with ht $J : I \geq \ell$, and that $I^{n+1} \subset JI^{\ell-g}$, for some integer $n \geq 1$. Then $I^{n+1} = JI^n$.*

Proof. Using Remark 2.1 this follows from [3, proof of 5.2]. □

Theorem 3.5. *Let R be a local Gorenstein ring of dimension d with infinite residue field, let I be an R-ideal of grade $g \geq 2$, analytic spread ℓ and minimal number of generators $n = \ell + 1$, and assume that I satisifes G_ℓ, that depth $R/I^j \geq d - g - j + 1$ for $1 \leq j \leq \ell - g + 1$, and that $I \subset I_1(\phi)^{\ell-g+2}$, where ϕ is a matrix with n rows presenting I. Then the following conditions are equivalent:*

(a) *After elementary row operations, the entries of one row generate $I_1(\phi)$.*

(b) *$r(I) \leq \ell - g + 1$.*

(c) *$\mathscr{R}$ is Cohen–Macaulay.*

Proof. Since I has second analytic deviation one, (a) and (b) are equivalent by [21, 5.1]. Now as I satisfies AN_ℓ^- by [20, 2.9], (b) implies (c) by Theorem 1.2. Then in particular G is Cohen–Macaulay [11, 1.1] and thus (c) implies (b) by Theorem 3.3. □

Naturally we can obtain stronger results by assuming the vanishing of the torsion of sufficiently many symmetric powers.

Corollary 3.6. *Let R be a local Cohen–Macaulay ring with infinite residue field, let I be an R-ideal with analytic spread ℓ, and minimal number of generators $n = \ell + 1$, let ϕ be a minimal presentation matrix, and assume that I satisfies G_ℓ, that G is Cohen–Macaulay and that $S_j(I) \cong I^j$ for $1 \leq j \leq \ell - g + 1$. Then*

(a) *$\mathscr{A}_{\ell-g+2} \neq 0$.*

(b) *If $I \subset I_1(\phi)^2$, then $r(I) = \ell - g + 1$.*

Proof. (a) Put $k = \max\{j \mid S_j(I) \cong I^j\}$. Since $k \geq \ell - g + 1$, I satisfies the assumptions of Theorem 3.3(a). But by the proof, setting $e = 1$ shows that $k \leq \ell - g + 1$. As for (b), by the assumption on the symmetric powers, $r(I) = 0$ or $r(I) \geq \ell - g + 1$. Since $n > \ell$, the result follows from Theorem 3.3. □

Remark 3.7. Under the assumptions of Theorem 3.3 (a) or (b), assume that $e \geq [(\ell - k)/(g-1) + 1]$ and that $g \geq 2$. Then $\mathscr{R}$ is Cohen–Macaulay.

Proof. Since I satisfies G_ℓ, it is enough to show that $r < \ell$ by [4, 5.1] or [17, 3.6]. But since $e = [(\ell - k)/(g-1) + 1] \geq 2$ by Corollary 3.6(a), the result follows from Theorem 3.3. □

The following theorem complements one of the main results of [17]. It applies immediately to grade 2 perfect ideals and grade 3 Gorenstein ideals satisfying G_ℓ and having second analytic deviation one.

Theorem 3.8. *Let R be a local Gorenstein ring with infinite residue field, let I be a strongly Cohen–Macaulay R-ideal of grade $g \geq 2$, analytic spread ℓ, and minimal number of generators $n = \ell + 1$ and assume that I satisfies G_ℓ, and that $I \subset I_1(\phi)^2$, where ϕ is a matrix with n rows presenting I. Then the following are equivalent:*

(a) *After elementary row operations, the entries of one row generate $I_1(\phi)$.*
(b) $r(I) \leq \ell - g + 1$.
(c) $rt(I) \leq \ell - g + 2$.
(d) $\mathscr{A}$ *is cyclic.*
(e) $\mathscr{R}$ *is Cohen–Macaulay.*
(f) G *is Cohen–Macaulay.*

Proof. By [17, 4.9], (a), (b), (c), and (d) are equivalent and each implies (e) (the latter also following from Theorem 1.2). One knows that (e) implies (f) by [11, 1.1], while now (f) implies (b) by Corollary 3.6(b). □

Remark 3.9 [21,2.11]. Under the assumptions of Theorem 3.8, let $s = v(I_1(\phi))$. Then type $\mathscr{R} = s + g - 2$ and type $G = s + 1$.

Remark 3.10. The condition $I \subset I_1(\phi)^2$ in Theorem 3.8 is really essential. A. Simis and B. Ulrich (unpublished) have discovered examples of strongly Cohen–Macaulay generically complete intersection prime ideals, with second deviation one in a Gorenstein ring (the diagonal ideal of a certain codimension 3 Gorenstein algebra) for which G is Cohen–Macaulay but $\mathscr{R}$ is not.

4. Analytically independent elements

One can use the results of the previous sections to say something about ideals having second analytic deviation zero, i.e. ideals generated by analytically independent elements. Recall that an ideal is *syzygetic* if $S_2(I) \cong I^2$; if I is generically a complete intersection, this is equivalent to the torsion-freeness of $H_1(I)$ over R/I.

Proposition 4.1. *Let R be a local Gorenstein ring of dimension d, and let I be an R-ideal of grade g and assume that I satisfies G_∞, that I is generated by n analytically independent elements and that G is Cohen–Macaulay. Then*

(a) $rt(I) \leq n - g$.

(b) *Assume further that depth* $R/I^j \geq d - g - j + 1$ *for* $1 \leq j \leq n - g - 2$. *Then* I *is of linear type if and only if* I *is syzygetic.*

Proof. We may assume the residue field is infinite. Part (a) follows from Lemma 2.2 and [18, 3.3] since $n = \ell(I)$. Now for (b) since G is Cohen–Macaulay, depth $R/I^{n-g-1} \geq d - \ell$ by [7, 3.3], and thus I satisfies $AN^-_{\ell-3}$ by [20, 2.9]. Since I is syzygetic, by part (a) it is enough to show $S_j(I) \cong I^j$ whenever $3 \leq j \leq n - g$. But since $I = J$ for any reduction J of I, this follows from Lemma 2.4 by induction on j. □

Corollary 4.2. *Let* R *be a local Gorenstein ring and let* I *be a Cohen–Macaulay ideal of deviation three satisfying* G_∞ *with* G *Cohen–Macaulay. Then* I *is of linear type if and only if* I *is syzygetic.*

Proof. It is enough to show that $n = \ell(I)$. But if $n = g + 3 > \ell$ then I would automatically satisfy $AN^-_{\ell-2}$ by e.g. [20, 2.9]. As I satisfies G_∞, it would follow that I has reduction number at most one [13, 4.7]. Since I is syzygetic, this would contradict the fact that $r(I) > 0$. □

G. Valla asked if a prime ideal in a regular local ring which is generated by analytically independent elements is necessarily of linear type. A counterexample was produced in [16]. It is a normal homogeneous Cohen–Macaulay prime ideal of grade 3 and deviation 3, in a polynomial ring over a field in 9 variables, which satisfies G_∞ and whose Rees algebra is Cohen–Macaulay. It follows by the result above that no such example can even be syzygetic. But Ulrich asked in [19] if a prime ideal is of linear type if the ideal is *locally* generated by analytically independent elements.

Theorem 4.3. *There exist homogeneous perfect prime ideals in* $k[x_1,\ldots,x_6]$ (*k an infinite field*) *of grade* 3 *and deviation* 3 *which are locally generated by analytically independent elements but are not of linear type.*

Proof. Take the above counterexample of [16] and specialize it by three general linear forms. This produces by Bertini's theorem the required prime ideal, in a six dimensional polynomial ring, which still satisfies G_∞, is generated by analytically independent elements and is not of linear type since the associated graded ring specializes [7]. The ideal is of deviation at most two on the punctured spectrum, hence is strongly Cohen–Macaulay [5] and hence of linear type [8, 9.1], locally on the punctured spectrum. □

Acknowledgements

I am grateful to B. Ulrich for helpful discussions and to the referee for a careful reading.

References

[1] I.M. Aberbach and S. Huckaba, Reduction number bounds on analytic deviation two ideals and Cohen–Macaulayness of associated graded rings, Comm. Algebra 23 (1995) 2003–2026.

[2] I.M. Aberbach, S. Huckaba and C. Huneke, Reduction numbers, Rees algebras and Pfaffian ideals, J. Pure Appl. Algebra 102 (1995) 1–15.

[3] I.M. Aberbach and C. Huneke, An improved Briançon-Skoda theorem with applications to the Cohen–Macaulayness of Rees algebras, Math. Ann. 297 (1993) 343–369.

[4] I.M. Aberbach, C. Huneke and N. V. Trung, Reduction numbers, Briançon-Skoda theorems and depth of Rees algebras, Comp. Math. 97 (1995) 403–434.

[5] L. Avramov and J. Herzog, The Koszul algebra of a codimension 2 embedding, Math. Z. 175 (1980) 249–280.

[6] D. Buchsbaum and D. Eisenbud, Algebraic structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math 99 (1977) 447–485.

[7] D. Eisenbud and C. Huneke, Cohen–Macaulay Rees algebras and their specializations, J. Alg. 81 (1983) 202–224.

[8] J. Herzog, A. Simis and W.V. Vasconcelos, Koszul homology and blowing-up rings, in: Commutative Algebra, Proceedings: Trento 1981, Lecture Notes in Pure and Applied Math., Vol. 84 (Marcel Dekker, New York, 1983) 79–169.

[9] J. Herzog, W.V. Vasconcelos and R. Villarreal, Ideals with sliding depth, Nagoya Math. J. 99 (1985) 159–172.

[10] S. Huckaba and C. Huneke, Powers of ideals having small analytic deviation, Amer. J. Math. 114 (1992) 367–403.

[11] C. Huneke, On the associated graded ring of an ideal, Ill. J. Math. 26 (1982) 121–137.

[12] C. Huneke and B. Ulrich, Residual intersections, J. Reine Angew. Math. 390 (1988) 1–20.

[13] M. Johnson and B. Ulrich, Artin–Nagata properties and Cohen–Macaulay associated graded rings, Comp. Math., to appear.

[14] S. Morey, Equations of blowups of ideals of codimension two and three, J. Pure Appl. Algebra 109 (1996) 197–211.

[15] D.G. Northcott and D. Rees, Reductions of ideals in local rings, Proc. Camb. Phil. Soc. 50 (1954) 145–158.

[16] A. Simis, B. Ulrich and W.V. Vasconcelos, Jacobian dual fibrations, Amer. J. Math. 115 (1993) 47–75.

[17] A. Simis, B. Ulrich and W.V. Vasconcelos, Cohen–Macaulay Rees algebras and degrees of polynomial relations, Math. Ann. 301 (1995) 421–444.

[18] N.V. Trung, Reduction exponent and degree bound for the defining equations of graded rings, Proc. Amer. Math. Soc. 101 (1987) 229–236.

[19] B. Ulrich, Remarks on residual intersections, in: Free Resolutions in Commuative Algebra and Algebraic Geometry, Research Notes in Mathematics, Vol. 2 (Jones and Bartlett, 1992) 133–138.

[20] B. Ulrich, Artin–Nagata properties and reductions of ideals, Contemp. Math. 159 (1994) 373–400.

[21] B. Ulrich, Ideals having the expected reduction number, Amer. J. Math. 118 (1996) 17–38.

[22] B. Ulrich and W.V. Vasconcelos, The equations of Rees algebras of ideals with linear presentation, Math. Z. 214 (1993) 79–92.

[23] W.V. Vasconcelos, On the equations of Rees algebras, J. Reine Angew. Math. 418 (1991) 189–218.